Written Exam for the M.Sc. in Economics summer 2011

Asset Pricing Theory

Final Exam

11 August 2011

(3-hour closed book exam)

Please note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by “eksamen på dansk” in brackets, you must write your exam paper in Danish.

If you are in doubt about which title you registered for, please see the print of your exam registration from the students’ self-service system.

The Exam consists of three problems each weighted by $33^1/_3\%$

Problem 1 — Market price of risk

We consider the properties of the financial derivatives f_1 and f_2 only depending on time and a single stochastic process V satisfying:

$$dV_t = aV_t dt + bV_t dz_t$$

where a and b > 0 are constants, and z is a standard Wiener process. Then we must have:

$$df_i(V_t, t) = \mu_i(V_t, t) f_i dt + \sigma_i(V_t, t) f_i dz_t \quad \text{for } i = 1,2.$$

(a) Construct a portfolio Π of the derivatives f_1 and f_2, such that Π is instantaneously riskless.

(b) Show that under the no arbitrage condition we must have:

$$\mu_1\sigma_2 - \mu_2\sigma_1 = r\sigma_2 - r\sigma_1$$

where r is the riskless interest rate.

(c) Introduce the market price of risk λ and explain why it does not depend on the nature of the derivatives. Determine the market price of risk if the investors are risk neutral.

(d) Show that any derivative f only depending on time and V necessarily satisfies:

$$f_t dz_t$$

where σ is the volatility of f and λ is the market price of risk.

Problem 2 — One factor models of the interest rate

The stochastic differential equation (SDE) for the short rate in the Vasiček model is:

$$dr_t = (b - ar_t)dt + \sigma dz_t$$

(a) Write the SDE for the short rate in the CIR-model.

(b) Explain the intuition of the drift term of the above SDEs and why that is appropriate to describe the process of the interest rate.

(c) Explain the difference in the volatility term of the two models, and what implications it has for the short rate in the two models.

(d) The Vasiček and the CIR model both give affine term structures, what does this mean.

(e) Find the expected value of r_t at time 0 (i.e. $E_0[r_t]$) in the Vasiček model

Hint: define $X_t = g(r_t, t) = e^{at} r_t$, and use Itô:

$$dg(r_t, t) = \left(\frac{\partial g(r_t, t)}{\partial t} + \frac{\partial g(r_t, t)}{\partial r_t}\mu(r_t, t) + \frac{1}{2}\frac{\partial^2 g(r_t, t)}{\partial r_t^2}\sigma(r_t, t)^2\right)dt + \frac{\partial g(r_t, t)}{\partial r_t}\sigma(r_t, t)dz_t$$

Where $dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dz_t$

(f) Show that the mean can be written as $E_0[r_t] = \alpha_t r_0 + (1 - \alpha_t) r_\infty$, and describe this equation.

(g) Find the variance of r_t at time 0

Hint: use this lemma

> Let $\sigma(t)$ be a deterministic function of time and define the process Y by
>
> $$dY_t = \int_0^t \sigma(s)\, dz_s$$
>
> Then Y_t has a normal distribution with zero mean and a variance given by
>
> $$Var[Y_t] = \int_0^t \sigma^2(s)\, ds$$

Problem 3 FX securities and two factor Itô

Suppose that the two FX-rates X_t and Y_t follows geometric Brownian motions.

You can interpret X as the USD amount one has to pay for one EUR and Y as USD amount one has to pay for one GBP

$$dX_t = \mu^X X_t dt + \sigma^X X_t dz_t^X$$

$$dY_t = \mu^Y Y_t dt + \sigma^Y Y_t dz_t^Y$$

Where z_t^X and z_t^Y are a standard Brownian motions (Wiener processes)

Suppose that we have the FX-rate V that can be interpreted as GBP amount one has to pay for one EUR

(a) Argue that V = G(X,Y) = X / Y

Itô's lemma for a function of to stochastic variables:

$$dG(X_t, Y_t, t) = \partial G(X_t, Y_t, t)/\partial t + \partial G(X_t, Y_t, t)/(\partial X_t)\, dX_t + \partial G(X_t, Y_t, t)/(\partial Y_t)\, dY_t + 1/2\ (\partial^2 \ldots$$

Where $dtdt = 0, dz_t^i dt = 0$, $dz_t^i dz_t^i = dt$ and $dz_t^X dz_t^Y = \rho dt$

(b) Show that V_t also follows a geometric Brownian motion, satisfying

$$dV_t = (\mu^X + \mu^Y + (\sigma^Y)^2 + \rho\sigma^X\sigma^Y)V_t dt + \sigma^X V_t dz_t^X - \sigma^Y V_t dz_t^Y$$

Now reformulate the stochastic differential equation for V to

$$dV_t = \mu^V V_t dt + \sigma^V V_t dz_t^V$$

Where $z_t^V = \alpha z_t^X + \beta z_t^Y$ with the condition $\alpha^2 + \beta^2 - 2\rho\ \alpha\beta = 1$

The constants are set to $\alpha = \dfrac{\sigma^X}{\sigma^Z}$ and $\beta = \dfrac{\sigma^Y}{\sigma^Z}$

(c) argue that z_t^V will be a standard Brownian motion (Wiener process)

Standard Brownian motion:

1) $z_0=0$.
2) The process z has independent increments, i.e. if $r < s \leq t < u$ then $z_u - z_t$ and $z_s - z_r$ are independent stochastic variables.
3) For $s < t$ the stochastic variable $z_t - z_s$ has the Gaussian distribution $N(0, \sqrt{t-s})$.
4) z is continuous.

It is more common to observe implied volatilities in the financial markets than implied correlations.

(d) Show that implied correlation ρ between X and Y can be described as a function of the volatilities σ^X, σ^Y and σ^V.

Assume you are a FX-option trader, and have some financial derivatives such that you got implied correlation risk. E.g. on the implied correlation between EURUSD and GBPEUR.

(e) With the results found above, describe how you can hedge your correlation risk, with FX-options.